

EQUATIONS IN OLIGOMORPHIC CLONES AND THE CONSTRAINT SATISFACTION PROBLEM FOR ω -CATEGORICAL STRUCTURES

LIBOR BARTO, MICHAEL KOMPATSCHER, MIROSLAV OLŠÁK, TRUNG VAN PHAM,
AND MICHAEL PINSKER

ABSTRACT. There exist two conjectures for constraint satisfaction problems (CSPs) of reducts of finitely bounded homogeneous structures: the first one states that tractability of the CSP of such a structure is, when the structure is a model-complete core, equivalent to its polymorphism clone satisfying a certain non-trivial linear identity modulo outer embeddings. The second conjecture, challenging the approach via model-complete cores by reflections, states that tractability is equivalent to the linear identities (without outer embeddings) satisfied by its polymorphisms clone, together with the natural uniformity on it, being non-trivial.

We prove that the identities satisfied in the polymorphism clone of a structure allow for conclusions about the orbit growth of its automorphism group, and apply this to show that the two conjectures are equivalent. We contrast this with a counterexample showing that ω -categoricity alone is insufficient to imply the equivalence of the two conditions above in a model-complete core.

Taking a different approach, we then show how the Ramsey property of a homogeneous structure can be utilized for obtaining a similar equivalence under different conditions.

We then prove that any polymorphism of sufficiently large arity which is totally symmetric modulo outer embeddings of a finitely bounded structure can be turned into a non-trivial system of linear identities, and obtain non-trivial linear identities for all tractable cases of reducts of the rational order, the random graph, and the random poset.

Finally, we provide a new and short proof, in the language of monoids, of the existence and uniqueness of the model-complete core of an ω -categorical structure.

1. INTRODUCTION

In order to keep the presentation of the wide topic of the present article as compact as possible, we postpone most definitions to an own preliminaries section (Section 2).

1.1. Constraint Satisfaction Problems. The Constraint Satisfaction Problem (CSP) of a structure \mathbb{A} in a finite relational language, denoted by $\text{CSP}(\mathbb{A})$, is the computational problem of deciding its primitive positive theory: given a sentence ϕ which is an existentially quantified conjunction of atomic formulas, decide whether or not ϕ holds in \mathbb{A} . When \mathbb{A} has a finite domain, then its CSP is in NP, and it has been conjectured that its CSP is always either NP-complete or polynomial-time solvable [FV99]. While the CSP of structures with an infinite domain can be of any complexity [BG08], and can in particular be undecidable, for a certain class of infinite-domain CSPs a similar dichotomy conjecture as for the finite case has been stated. In fact, two such conjectures have been brought up via different approaches; in the

Date: January 11, 2017.

Libor Barto and Miroslav Olšák were supported by the the Grant Agency of the Czech Republic, grant GAČR 13-01832S. The research of Michael Kompatscher, Trung Van Pham, and Michael Pinsker has been funded through project P27600 of the Austrian Science Fund (FWF).

present article we first establish their equivalence, and then investigate in more detail the tractability conditions of the two conjectures.

The range of both conjectures are reducts of finitely bounded homogeneous structures, a (proper) subclass of the countable ω -categorical structures. It is well-known, and easy to see from the definition, that the CSP of such structures is contained in NP; both conjectures state that it is always either NP-complete or contained in P, but each conjecture gives a different delineation between the (NP-)hard and the tractable (i.e., polynomial-time solvable) cases.

1.2. The first conjecture. The first and older conjecture, formulated by Bodirsky and Pinsker (cf. [BPPa]), is based on the notion of the model-complete core of an ω -categorical structure, which can be viewed as the simplest representative in the class of an ω -categorical structure with respect to the equivalence relation of homomorphic equivalence. We have the following.

Theorem 1.1 (Bodirsky [Bod07]). *Every countable ω -categorical structure is homomorphically equivalent to a model-complete core. This model-complete core is unique up to isomorphism and itself ω -categorical.*

The idea leading to the first conjecture is that the complexity of the CSP of a structure \mathbb{A} in the range of the conjecture is determined by which finite structures \mathbb{C} have a primitive positive (pp-) interpretation with parameters in its model-complete core \mathbb{B} . This approach builds on two facts: the first fact being that homomorphically equivalent structures have the same primitive positive theory, and hence \mathbb{A} and \mathbb{B} have equal CSPs; and the second fact being that if a structure \mathbb{C} has a primitive positive interpretation with parameters in an ω -categorical model-complete core \mathbb{B} , then $\text{CSP}(\mathbb{C})$ reduces to $\text{CSP}(\mathbb{B})$ in polynomial time. It is a well-known fact that the structure

$$\mathbb{S} := (\{0, 1\}; \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\})$$

pp-interprets all finite structures, and that its CSP is NP-complete.

Conjecture 1.2. *Let \mathbb{A} be a reduct of a finitely bounded homogeneous structure, and let \mathbb{B} be its model-complete core. Then one of the following holds.*

- (i) \mathbb{B} pp-interprets \mathbb{S} with parameters (and consequently, $\text{CSP}(\mathbb{A})$ is NP-complete).
- (ii) $\text{CSP}(\mathbb{A})$ is polynomial-time solvable.

From our remarks above it follows that if condition (i) in Conjecture 1.2 holds, then $\text{CSP}(\mathbb{A})$ is indeed NP-complete. What remains to prove is that if this condition is not satisfied, then $\text{CSP}(\mathbb{A})$ is tractable. The following equivalent conditions have been established for this situation via the polymorphism clone $\text{Pol}(\mathbb{B})$ of a structure \mathbb{B} ((ii) in [BP15b], and (iii), (iv) in [BP16a]). We denote the clone of projections on the set $\{0, 1\}$ by $\mathbf{1}$; then $\mathbf{1} = \text{Pol}(\mathbb{S})$.

Theorem 1.3. *Let \mathbb{B} be an ω -categorical model-complete core. The following are equivalent.*

- (i) \mathbb{B} does not pp-interpret \mathbb{S} with parameters.
- (ii) No stabilizer of $\text{Pol}(\mathbb{B})$ maps homomorphically and continuously to $\mathbf{1}$.
- (iii) No stabilizer of $\text{Pol}(\mathbb{B})$ maps homomorphically to $\mathbf{1}$.
- (iv) $\text{Pol}(\mathbb{B})$ has a Siggers term modulo outer embeddings, i.e., there exist $e_1, e_2, f \in \text{Pol}(\mathbb{B})$ such that the identity

$$e_1 \circ f(x, y, x, z, y, z) = e_2 \circ f(y, x, z, x, z, y)$$

holds in $\text{Pol}(\mathbb{B})$.

Observe that the very recent condition (iv) turns, for the first time, the supposed tractability criterion of Conjecture 1.2 into a positive statement, nourishing the hope for a positive answer to the conjecture.

1.3. The second conjecture. The second and younger conjecture was born from the observation that the usage of homomorphic equivalence and pp-interpretations might not be optimal in the order which leads to Conjecture 1.2, as the crucial structure \mathbb{S} might, for example, be homomorphically equivalent to a structure with a pp-interpretation in \mathbb{A} , but not pp-interpretable with parameters by the model-complete core of \mathbb{A} . This suggests the following weaker conjecture, which does use the reductions by homomorphic equivalence and pp-interpretations in the best possible way [BOP].

Conjecture 1.4. *Let \mathbb{A} be a reduct of a finitely bounded homogeneous structure. Then one of the following holds.*

- (i) \mathbb{S} is homomorphically equivalent to a structure with a pp-interpretation in \mathbb{A} (and consequently, $\text{CSP}(\mathbb{A})$ is NP-complete).
- (ii) $\text{CSP}(\mathbb{A})$ is polynomial-time solvable.

It has been remarked in [BOP] that the two conjectures are equivalent for finite structures. While more likely to be true, one disadvantage of Conjecture 1.4 is that there is no unique optimal structure that can be pp-interpreted in \mathbb{A} , as opposed to the model-complete core for homomorphic equivalence. Similarly to (ii) in Theorem 1.3, the authors of [BOP] did however provide an equivalent tractability criterion using identities and topology.

Theorem 1.5. *Let \mathbb{A} be ω -categorical. The following are equivalent.*

- (i) \mathbb{S} is not homomorphically equivalent to a structure with a pp-interpretation in \mathbb{A} .
- (ii) $\text{Pol}(\mathbb{A})$ does not have a uniformly continuous h1 clone homomorphism to $\mathbf{1}$.

Note that a positive statement equivalent to the statements of Theorem 1.5, i.e., an analogue of (iv) in Theorem 1.3 is missing, leaving Conjecture 1.4 somewhat less accessible than Conjecture 1.2.

1.4. Equivalence of the conjectures. The results known so far concerning identities in polymorphism clones were shown for all ω -categorical model-complete cores, rather than for the considerably more restricted class of structures concerned by the conjectures; it is probably fair to say that it seemed inconceivable that assumptions like finite boundedness would be useful when proving such structural results (these assumptions are, however, essential for the algorithmic aspects of the CSPs). Therefore, the most likely way of showing the equivalence of the conjectures seemed by proving that for all ω -categorical model-complete cores, conditions (iv) in Theorem 1.3 and (ii) in Theorem 1.5 are equivalent: that is, since the other implication is well-known and easy, that a Siggers term modulo outer embeddings prevents a uniformly continuous h1 clone homomorphism to $\mathbf{1}$.

We will, however, provide a counterexample, basically the atomless Boolean algebra with the right choice of relations, showing that this is not true in general.

Theorem 1.6. *There exists an ω -categorical model-complete core structure \mathbb{A} such that:*

- (i) No stabilizer of $\text{Pol}(\mathbb{A})$ has a continuous clone homomorphism to $\mathbf{1}$ (and hence, by Theorem 1.3, $\text{Pol}(\mathbb{A})$ has a Siggers term modulo outer embeddings).
- (ii) $\text{Pol}(\mathbb{A})$ has a uniformly continuous h1 clone homomorphism to $\mathbf{1}$.

Surprisingly, on the other hand, it turns out that every structure \mathbb{A} which is a counterexample as above must have at least double exponential orbit growth. This is remarkable in that it is the first instance discovered where structural higher-arity information about the polymorphism clone of an ω -categorical structure yields information about its automorphism group.

Theorem 1.7. *Let \mathbb{A} be a structure with the properties stated in Theorem 1.6. Then its automorphism group $\text{Aut}(\mathbb{A})$ must have at least double exponential orbit growth.*

From this, it is straightforward to derive the equivalence of the CSP conjectures, answering Problem 8.3 in [BOP] to the positive, and showing that the implication from (4) to (3) in Corollary 5.3 of [BP16a] holds.

Corollary 1.8. *Let \mathbb{A} be a reduct of a structure which is homogeneous in a finite relational language, and let \mathbb{B} be its model-complete core. Then the following are equivalent.*

- (i) *Some stabilizer of $\text{Pol}(\mathbb{B})$ has a continuous clone homomorphism to $\mathbf{1}$.*
- (ii) *$\text{Pol}(\mathbb{A})$ has a uniformly continuous h1 clone homomorphism to $\mathbf{1}$.*
- (iii) *$\text{Pol}(\mathbb{B})$ has a uniformly continuous h1 clone homomorphism to $\mathbf{1}$.*

In particular, Conjecture 1.2 holds if and only if Conjecture 1.4 holds.

1.5. The Ramsey property. Via an alternative approach involving Ramsey theory, we will then show a statement similar to Corollary 1.8 under different, and incomparable, conditions. Although this might seem irrelevant for CSPs considering our results above which cover the entire range of Conjectures 1.2 and 1.4, it is interesting that various conditions of very different nature seem to imply this statement, while at the same time we know from our counterexample in Theorem 1.6 that ω -categoricity alone is not sufficient. Observe that in the following theorem, there is no requirement of finite language, or orbit growth, or even ω -categoricity; on the other hand, we require the non-trivial linear identities to be satisfied modulo embeddings of an ordered Ramsey structure.

Theorem 1.9. *Let \mathbb{A} be a reduct of an ordered homogeneous Ramsey structure \mathbb{D} . If $\text{Pol}(\mathbb{A})$ satisfies a non-trivial set of linear identities modulo outer embeddings of \mathbb{D} , then $\text{Pol}(\mathbb{A})$ does not have a uniformly continuous h1 clone homomorphism to $\mathbf{1}$.*

Note that Theorem 1.9 corresponds to the contrapositive of the non-trivial implication from (iii) to (i) in Corollary 1.8, via the fact that (i) there is equivalent to the existence of a Siggers term modulo outer embeddings.

We would also like to remark that the situation of Theorem 1.9 is particularly interesting for the approach to Conjectures 1.2 and 1.4 via canonical functions, as surveyed in [BP11] (cf. also the recent [BP16b]); indeed, many of the successful CSP classifications via that approach yield tractable situations as in Theorem 1.9.

1.6. Linearization. Corollary 1.8, combined with Theorem 1.3, implies that if an ω -categorical model-complete core \mathbb{B} has a Siggers polymorphism modulo outer embeddings, then $\text{Pol}(\mathbb{B})$ does not have a uniformly continuous h1 clone homomorphism onto $\mathbf{1}$. It does not imply that in that situation, $\text{Pol}(\mathbb{B})$ satisfies non-trivial linear identities, i.e., that $\text{Pol}(\mathbb{B})$ does not have an h1 clone homomorphism to $\mathbf{1}$ disregarding the uniformity on $\text{Pol}(\mathbb{B})$. The situation in Theorem 1.9 is similar. It is hitherto unknown under which conditions non-trivial linear identities modulo outer embeddings imply non-trivial linear identities in a polymorphism clone; as of today, we cannot even refute the possibility that the existence of an h1 homomorphism

to **1** implies the existence of a uniformly continuous such homomorphism in general. This question, for ω -categorical model-complete cores, corresponds to the implication from (6) to (4) in [BP16a].

Approaching this problem, we are going to show that under the assumption of finite boundedness, and stronger identities than the Siggers identity modulo outer embeddings, we can derive the satisfaction of non-trivial linear identities in a polymorphism clone.

Theorem 1.10. *Let \mathbb{A} be a reduct of a finitely bounded homogeneous structure \mathbb{D} which is given by a set of forbidden substructures all of which have size at most $k \geq 3$. If $\text{Pol}(\mathbb{A})$ contains a k -ary polymorphism f which is totally symmetric modulo outer embeddings of \mathbb{D} , i.e., for all permutations ρ of $\{1, \dots, k\}$ satisfies an identity of the form*

$$e_{1,\rho} \circ f(x_1, \dots, x_k) = e_{2,\rho} \circ f(x_{\rho(1)}, \dots, x_{\rho(k)}),$$

where $e_{1,\rho}, e_{2,\rho} \in \overline{\text{Aut}(\mathbb{D})}$, then $\text{Pol}(\mathbb{A})$ does not have an h1 clone homomorphism to **1**.

We are further going to show that under different conditions, trading finite boundedness and total symmetry for near unanimity and strong preservation of the relations of \mathbb{D} , non-trivial linear identities can be derived as well.

Theorem 1.11. *Let \mathbb{D} be a structure which is homogeneous in a finite relational language. Suppose that f is a k -ary operation, where $k \geq 2$, satisfying the following two conditions:*

- *f is a strong polymorphism of \mathbb{D} , i.e., preserves all relations of \mathbb{D} and their negations;*
- *f is a near unanimity (nu) function modulo outer embeddings of \mathbb{D} , i.e., satisfies identities of the form*

$$\begin{aligned} e(x) &= e_1 \circ f(y, x, \dots, x) = e_2 \circ f(x, y, x, \dots, x) = \dots \\ &= e_k \circ f(x, \dots, x, y), \end{aligned}$$

where $e, e_1, \dots, e_k \in \overline{\text{Aut}(\mathbb{D})}$.

Then no reduct of \mathbb{D} with the polymorphism f has an h1 clone homomorphism to **1**.

From Theorems 1.10 and 1.11 and the classifications in [BK08], [BK09], [BP15a], and [KP] it follows directly that most reducts of the rationals, the random graph, and the random partial order with tractable CSPs have a polymorphism clone satisfying non-trivial linear identities. Using a similar proof technique for the remaining cases we obtain the following.

Theorem 1.12. *Let \mathbb{A} be a reduct of one of the following structures:*

- $(\mathbb{N}; =)$;
- the order $(\mathbb{Q}; \leq)$ of the rational numbers;
- the random partial order;
- the random graph.

Then $\text{Pol}(\mathbb{A})$ has a uniformly continuous h1 clone homomorphism to **1** if and only if it has an h1 clone homomorphism to **1**. When \mathbb{A} has a finite language, then its CSP is tractable if and only if $\text{Pol}(\mathbb{A})$ satisfies a non-trivial set of linear identities.

1.7. Cores. Theorem 1.1 above stating the existence and uniqueness of the model-complete core of an ω -categorical structure is of central importance for Conjecture 1.2, and calculating the model-complete core of structures has been an integral part of the major successful CSP classifications so far. While the alternative more recent Conjecture 1.4 threatened to make the

notion obsolete for its context, the equivalence of the conjectures established in the present article provides further evidence of the decisive role of model-complete cores for CSPs.

Observe that the notion of a model-complete core is defined via the endomorphism monoid of a structure (density of the invertibles in the monoid), so in particular structures with isomorphic (as topological monoids, cf. [BPPb, BEKP]) endomorphism monoids are either both model-complete cores, or none of them is. Moreover, by the theorem of Ryll-Nardzewski, Engeler, and Svenonius [Hod97], the condition of ω -categoricity of a countable structure is equivalent to oligomorphicity of its automorphism group, again captured by its endomorphism monoid. It thus seems natural to have a proof of Theorem 1.1 in the language of transformation monoids, without reference to the particular language of a structure. The original and quite lengthy proof due to Bodirsky, however, does work with structures, and it is not obvious how to translate it into a proof via monoids.

We shall provide a new, short proof of Theorem 1.1 using topological monoids, which perhaps reflects better the combinatorial content of the theorem, and in particular connects it to the recent notion of reflections (which in turn leads to the other conjecture, Conjecture 1.4). Set naturally in the language of monoids, our proof yields simultaneously the generalization of the theorem to weakly oligomorphic structures given in [PP16].

1.8. Organization of this article. We provide definitions and notation in Section 2. The main results about the two CSP conjectures, Theorems 1.6 and 1.7, Corollary 1.8, and Theorem 1.9, are shown in Section 3. In Section 4, we investigate the relationship between linear identities modulo outer embeddings and those without outer embeddings, proving Theorems 1.10, 1.11, and 1.12. The new proof of Theorem 1.1, and new insights connecting it directly to reflections, are provided in Section 5.

2. PRELIMINARIES

We explain the notions which appeared in the introduction, and fix some notation for the rest of the article. For undefined universal algebraic concepts and more detailed presentations of the notions presented here we refer to [BS81, Ber11]. For notions from model theory we refer to [Hod97].

2.1. Polymorphism clones, automorphisms, and invertibles. We denote relational structures by \mathbb{A}, \mathbb{B} , etc. When \mathbb{A} is a relational structure, we reserve the symbol A for its domain. We write $\text{Pol}(\mathbb{A})$ for its *polymorphism clone*, i.e., the set of all finitary operations on A which preserve all relations of \mathbb{A} . The polymorphism clone $\text{Pol}(\mathbb{A})$ is always a *function clone*, i.e., it is closed under composition and contains all projections. The unary functions in $\text{Pol}(\mathbb{A})$ are precisely the *endomorphisms* of \mathbb{A} , denoted by $\text{End}(\mathbb{A})$. The endomorphisms which are bijections and whose inverse function is also an endomorphism are precisely the *automorphisms* of \mathbb{A} . We denote the set of automorphisms of \mathbb{A} by $\text{Aut}(\mathbb{A})$.

When \mathcal{A} is any function clone (not necessarily the polymorphism clone of a structure), then still the unary functions in \mathcal{A} form a transformation monoid, and the unary invertible functions in \mathcal{A} (i.e., those having an inverse in \mathcal{A}) form a permutation group, the *group of invertibles of \mathcal{A}* . We write A for the domain of the function clone \mathcal{A} .

2.2. Clone homomorphisms. A *clone homomorphism* from a function clone \mathcal{A} to a function clone \mathcal{B} is a mapping $\xi: \mathcal{A} \rightarrow \mathcal{B}$ which

- preserves arities, i.e., it sends every function in \mathcal{A} to a function of the same arity in \mathcal{B} ;

- preserves each projection, i.e., it sends the k -ary projection onto the i -th coordinate in \mathcal{A} to the same projection in \mathcal{B} , for all $1 \leq i \leq k$;
- preserves composition, i.e., $\xi(f(g_1, \dots, g_n)) = \xi(f)(\xi(g_1), \dots, \xi(g_n))$ for all n -ary functions f and all m -ary functions g_1, \dots, g_n in \mathcal{A} .

For all $1 \leq i \leq k$ we denote the k -ary projection onto the i -th coordinate by π_i^k , in any function clone and irrespectively of the domain of that clone. This slight abuse of notation allows us, for example, to express the second item above by writing $\xi(\pi_i^k) = \pi_i^k$.

A mapping $\xi: \mathcal{A} \rightarrow \mathcal{B}$ is called an *h1 clone homomorphism* if it preserves arities and composition with projections, i.e., $\xi(f(g_1, \dots, g_n)) = \xi(f)(\xi(g_1), \dots, \xi(g_n))$ for all n -ary functions f in \mathcal{A} and all m -ary projections g_1, \dots, g_n . If, in addition, ξ preserves each projection, then it is called a *strong h1 clone homomorphism*. Note that an h1 clone homomorphism to $\mathbf{1}$ is automatically strong.

2.3. Identities / Equations. The clone homomorphisms are those mappings between clones preserving *identities*, i.e., universally quantified equations between terms built from the functions in clones (with an appropriate language providing a symbol for every element of the clones). The h1 clone homomorphisms are those mappings between clones preserving identities of height one, and strong h1 clone homomorphisms preserve all identities of height at most one, also known as *linear identities*, i.e., identities where no nesting of functions is allowed; cf. [BOP]. A *linear identity modulo outer unary functions* is a universally quantified equation of the form $e_1 \circ s = e_2 \circ t$, where e_1, e_2 are unary and s, t are terms of height at most one.

When \mathbb{A} is a relational structure, then a *linear identity of $\text{Pol}(\mathbb{A})$ modulo outside endomorphisms (automorphisms, embeddings)* is an identity of the form $e_1 \circ s = e_2 \circ t$ which holds in $\text{Pol}(\mathbb{A})$, where $e_1, e_2 \in \text{Pol}(\mathbb{A})$ are endomorphisms (automorphisms, embeddings) of \mathbb{A} , and s, t terms over $\text{Pol}(\mathbb{A})$ of height at most one. Similarly, we speak of linear identities modulo outside embeddings (automorphisms, embeddings) of \mathbb{D} , where \mathbb{D} is some other structure, with the obvious meaning.

A set of identities is *non-trivial* if it is unsatisfiable in the clone $\mathbf{1}$. Therefore, a function clone satisfies a non-trivial set of identities if and only if it does not have a clone homomorphism to $\mathbf{1}$; it satisfies a non-trivial set of linear identities if and only if it does not have an h1 clone homomorphism to $\mathbf{1}$. It follows from the compactness theorem of first-order logic that these non-trivial sets of identities can be chosen to be finite.

2.4. Stabilizers. When \mathcal{A} is a function clone, and $F \subseteq A$ is a finite subset of its domain, then the (pointwise) *stabilizer* of F in \mathcal{A} , denoted by \mathcal{A}_F , is the function clone of all $f \in \mathcal{A}$ satisfying $f(a, \dots, a) = a$ for all $a \in F$. We emphasize that we always understand stabilizers to be pointwise, and always of a finite set.

We remark that when \mathbb{A} is a relational structure and $F \subseteq A$ is finite, then the stabilizer of F in $\text{Pol}(\mathbb{A})$ is the polymorphism clone of the structure obtained by enriching \mathbb{A} by a unary singleton relation $\{a\}$ for every $a \in F$.

2.5. Topology. Every function clone is naturally equipped with the topology of pointwise convergence: in this topology, a sequence $(f_i)_{i \in \omega}$ of n -ary functions converges to an n -ary function f on the same domain if and only if for all n -tuples \bar{a} of the domain the functions f_i agree with f on \bar{a} for all but finitely many $i \in \omega$. Therefore, every function clone gives rise to an abstract *topological clone* which reflects this topology as well as the composition structure of the clone [BPPb].

We always imagine function clones to carry the pointwise convergence topology, which is, in the case of a countable domain, in fact induced by a metric, and in general by a uniformity [BPPb, GP16, Sch15]. Then a mapping $\xi: \mathcal{A} \rightarrow \mathcal{B}$, where \mathcal{A} and \mathcal{B} are function clones, is continuous if and only if for all $f \in \mathcal{A}$ and all finite sets $B' \subseteq B$ there exists a finite set $A' \subseteq A$ such that for all $g \in \mathcal{A}$ of the same arity as f , if g agrees with f on A' , then $\xi(g)$ agrees with $\xi(f)$ on B' . It is uniformly continuous if and only if for all $n \geq 1$ and all finite $B' \subseteq B$ there exists a finite $A' \subseteq A$ such that whenever two n -ary functions $f, g \in \mathcal{A}$ agree on A' , then their images $\xi(f), \xi(g)$ agree on B' . Note that in the case of mappings $\xi: \mathcal{A} \rightarrow \mathbf{1}$, uniform continuity means that for every $n \geq 1$ there exists a finite $A' \subseteq A$ such that $\xi(f)$ only depends on the restriction of f to A' , for all n -ary $f \in \mathcal{A}$. When ξ is an h1 clone homomorphism, then A' can be chosen independently of n .

We remark that the polymorphism clones of relational structures are precisely the function clones which are complete with respect to this uniformity (or, put differently, closed in the function clone of all functions of the domain). Function clones on a finite domain are discrete.

2.6. Oligomorphicity, ω -categoricity and orbit growth. Recall that by the theorem of Ryll-Nardzewski, Engeler, and Svenonius, a countable relational structure \mathbb{A} is ω -categorical if and only if its automorphism group $\text{Aut}(\mathbb{A})$ is *oligomorphic*, i.e., for every $n \geq 1$, the natural componentwise action of $\text{Aut}(\mathbb{A})$ on A^n has only finitely many orbits. In particular finite structures are always ω -categorical. Every countable ω -categorical structure \mathbb{A} thus induces a monotone function on the positive natural numbers which assigns to every $n \geq 1$ the number of orbits of n -tuples with respect to $\text{Aut}(\mathbb{A})$; we call this function the *orbit growth* of \mathbb{A} (or of $\text{Aut}(\mathbb{A})$). There exist ω -categorical structures of arbitrarily fast orbit growth.

Similarly, we say that a function clone is *oligomorphic* if its group of unary invertibles is, and we can hence naturally speak of the orbit growth of an oligomorphic function clone.

2.7. Homogeneity, finite boundedness, and the Ramsey property. The ω -categorical structures concerned by the conjectures above are reducts of finitely bounded homogeneous structures. Here, following [Tho91] and numerous subsequent authors, we define a *reduct* of a relational structure \mathbb{A} to be a relational structure on the same domain all of whose relations have a first-order definition in \mathbb{A} without parameters.

A relational structure is *homogeneous* if every partial isomorphism between finite substructures extends to an automorphism of the entire structure. A countable relational structure \mathbb{A} is *finitely bounded* if it has a finite signature, and there exists a finite set F of finite structures in its signature such that \mathbb{A} contains precisely those structures as induced substructures which embed no member of F . We are going to call every such F a set of *forbidden substructures* (with respect to \mathbb{A}).

A relational structure \mathbb{D} is *Ramsey* if for all finite induced substructures \mathbb{P}, \mathbb{Q} of \mathbb{D} and all functions χ from the isomorphic copies of \mathbb{P} in \mathbb{D} to $\{0, 1\}$ there exists an isomorphic copy of \mathbb{Q} in \mathbb{D} on which χ is constant. It is *ordered* if it first-order defines (without parameters) a linear order on its domain. For more details about Ramsey structures in this context, we refer to the surveys [BP11], [Bod12].

2.8. Homomorphic equivalence and model-complete cores. When relational structures \mathbb{A} and \mathbb{B} have the same signature, then we say that \mathbb{A} and \mathbb{B} are *homomorphically equivalent* if there exists a homomorphism $\mathbb{A} \rightarrow \mathbb{B}$ and a homomorphism $\mathbb{B} \rightarrow \mathbb{A}$. A relational structure \mathbb{B} is called a *model-complete core* if $\text{Aut}(\mathbb{B})$ is dense in $\text{End}(\mathbb{B})$, i.e., for every endomorphism e of \mathbb{B} and every finite subset B' of B there exists an automorphism of \mathbb{B}

which agrees with e on B' . When \mathbb{B} is finite, then this means that every endomorphism is an automorphism, and \mathbb{B} is simply called a *core*.

Similarly, we call a function clone or a transformation monoid a *model-complete core* if the group of its invertible functions is dense in its unary functions.

2.9. CSPs. For a finite relational signature Σ and a Σ -structure \mathbb{A} , the *constraint satisfaction problem* of \mathbb{A} , or $\text{CSP}(\mathbb{A})$ for short, is the membership problem for the class

$$\{\mathbb{C} \mid \mathbb{C} \text{ is a finite } \Sigma\text{-structure and} \\ \text{there exists a homomorphism } \mathbb{C} \rightarrow \mathbb{A}\}.$$

An alternative definition of $\text{CSP}(\mathbb{A})$ is via primitive positive (pp-) sentences. Recall that a *pp-formula* over \mathbb{A} is a first order formula which only uses predicates from \mathbb{A} , conjunction, equality, and existential quantification. $\text{CSP}(\mathbb{A})$ can equivalently be phrased as the membership problem of the set of pp-sentences which are true in \mathbb{A} .

3. EQUIVALENCE OF THE CONJECTURES, AND THE RAMSEY PROPERTY

This section is divided into three parts: we first prove Theorem 1.7 and Corollary 1.8 in Section 3.1, and then provide the counterexample of Theorem 1.6 in Section 3.2. Finally, we turn to applications of the Ramsey property in Section 3.3, proving Theorem 1.9.

3.1. Orbit Growth and Equivalence of the conjectures.

Definition 3.1. Let \mathcal{C} be a function clone, and let $S \subseteq C$ be a subset of its domain with $|S| \geq 2$. Then a function $g: C \rightarrow S$ is a *retractional witness for \mathcal{C} with respect to S* if the restriction of $g \circ t$ to S is a projection on S for all $t \in \mathcal{C}$.

For an n -ary $t \in \mathcal{C}$, we call an index $1 \leq i \leq n$ *fundamental for t with respect to S* if there exists a retractional witness g for \mathcal{C} with respect to S such that $g \circ t|_{S^n}$ is the i -th n -ary projection π_i^n on S .

The *ambiguity degree of $t \in \mathcal{C}$ with respect to S* is the number of its fundamental indices with respect to S . The ambiguity degree of \mathcal{C} is the supremum of the ambiguity degrees of its members with respect to sets $S \subseteq C$ of at least two elements:

$$\sup \{d \in \omega \mid \exists t \in \mathcal{C}, S \subseteq C (|S| \geq 2 \wedge t \text{ has ambiguity} \\ \text{degree } d \text{ with respect to } S)\}$$

Lemma 3.2. *Let \mathcal{C} be a function clone of infinite ambiguity degree. Then the componentwise action of the group of unary invertible functions in \mathcal{C} on C^n has at least $2^{2^n} - 1$ orbits, for all $n \geq 1$.*

Proof. Given $n \geq 1$, pick $t(x_1, \dots, x_k) \in \mathcal{C}$ of ambiguity degree at least 2^n with respect to $S \subseteq C$ of at least two elements; by taking a subset, we may assume $|S| = 2$. By identifying some variables of t with variables corresponding to a fundamental index of t , we may assume that all indices of t are fundamental, and that $k = 2^n$. For any non-empty subset R of S^n , pick an n -tuple $q^R \in C^n$ of the form $t(q_1^R, \dots, q_k^R)$ (applied componentwise), where every $q_i^R \in R$, and all tuples in R appear as some q_i^R .

We claim that when $R \neq R'$, then q^R and $q^{R'}$ lie in distinct orbits. To see this, suppose without loss of generality that $R \setminus R' \neq \emptyset$. Thus there exists some $q_i^R \notin R'$. Let $g: C \rightarrow S$ be the retractional witness such that $g \circ t|_{S^k}$ projects to the i -th coordinate. If $q^R = \alpha(q^{R'})$ for an invertible $\alpha \in \mathcal{C}$, then we would have $g(q^R) = g \circ \alpha(q^{R'}) = g \circ \alpha \circ t(q_1^{R'}, \dots, q_k^{R'})$. Observe

that $g \circ (\alpha \circ t)|_{S^*}$ is a projection since $\alpha \circ t \in \mathcal{C}$ and since g is a retractional witness. Hence, $q_i^R = g(q^R) \in \{q_1^{R'}, \dots, q_k^{R'}\}$, a contradiction. \square

Lemma 3.3. *Let \mathcal{C} be a function clone which is a model-complete core and which satisfies some non-trivial linear identity modulo outer unary functions. If \mathcal{C} has a retractional witness, then it has infinite ambiguity degree.*

Proof. For any $t \in \mathcal{C}$ of ambiguity degree $n \geq 1$, we find $t' \in \mathcal{C}$ of ambiguity degree $2n$. So let $t \in \mathcal{C}$ be given, and let $S \subseteq C$ be a 2-element set such that t has n fundamental indices with respect to S , witnessed by functions $g_1, \dots, g_n: C \rightarrow S$. By identifying variables we may assume that t is n -ary. Renaming the variables, we may further assume that g_i witnesses the index i , for $1 \leq i \leq n$. Set $R := t[S^n]$. Because \mathcal{C} is a model-complete core, in the stabilizer \mathcal{C}_R a nontrivial identity which is linear modulo outer unary functions is satisfied, since linear identities modulo outer functions which hold in a model-complete core also hold in all of its stabilizers (this is easy to see and well-known, but we refer to [BP16a]). Let $s \in \mathcal{C}_R$ witness this, i.e., s satisfies the nontrivial identity $u \circ s(y_1, \dots, y_m) = v \circ s(z_1, \dots, z_m)$, for variables $y_1, \dots, y_m, z_1, \dots, z_m$ which are not necessarily distinct. We claim that the nm -ary term

$$s * t := s(t(x_1^1, \dots, x_n^1), \dots, t(x_1^m, \dots, x_n^m))$$

has the desired property.

To see this, we first observe that $g_i \circ (s * t)|_{S^{nm}}$ is a projection, and in fact a projection onto a variable of the form x_i^j : inserting variables x_1, \dots, x_n into $s * t$, we obtain

$$\begin{aligned} g_i \circ (s * t)(x_1, \dots, x_n, \dots, x_1, \dots, x_n) &|_{S^n} \\ &= g_i \circ s(t(x_1, \dots, x_n), \dots, t(x_1, \dots, x_n))|_{S^n} \\ &= g_i \circ t(x_1, \dots, x_n)|_{S^n} \\ &= \pi_i^n(x_1, \dots, x_n)|_{S^n}, \end{aligned}$$

with the second equation holding since R is stabilized by s . In particular, $s * t$ has ambiguity degree at least n , witnessed by g_1, \dots, g_n . Note furthermore that for the same reason, the functions $u \circ (s * t)$ and $v \circ (s * t)$ have ambiguity degree at least n , projecting to a variable of the form x_i^j when composed with g_i from the left. This can be restated by saying that $s * t$ has ambiguity degree at least n , with fundamental indices corresponding to variables of the form x_i^j witnessed by two witnesses $g_i \circ u$ and $g_i \circ v$; we now argue that the fundamental indices witnessed by $g_i \circ u$ and $g_i \circ v$ are distinct.

To this end, fix any $1 \leq i \leq n$, and say that $g_i \circ u \circ (s * t)|_{S^{nm}}$ projects onto x_i^j , where $1 \leq j \leq m$. Since the equation $u \circ s(y_1, \dots, y_m) = v \circ s(z_1, \dots, z_m)$ is non-trivial, we must have $y_j \neq z_j$. On the other hand, we must have that $y_j \in \{z_1, \dots, z_m\}$, since s obviously depends on its j -th variable. Thus $y_j = z_\ell$ for some $\ell \neq j$. This means that $g_i \circ v$ is a retractional witness such that $(g_i \circ v) \circ (s * t)|_{S^{nm}}$ projects onto the variable with index x_i^ℓ , proving our claim.

Summarizing, each fundamental index of t , witnessed by some g_i , has a corresponding fundamental index of $s * t$, also witnessed by g_i , and this assignment is injective; and moreover, each fundamental index of $s * t$, witnessed by g_i , yields two fundamental indices of $s * t$, witnessed by $g_i \circ u$ and $g_i \circ v$, respectively. \square

We thus obtain the following theorem, which shows, in particular, how equational properties of the polymorphism clone of a structure can have implications about its automorphism group. We first formulate it in terms of function clones, and then restate it in terms of structures.

Theorem 3.4. *Let \mathcal{C} be an oligomorphic function clone which is a model-complete core. Suppose that*

- (i) \mathcal{C} satisfies a non-trivial linear identity modulo outer unary functions, and
- (ii) \mathcal{C} has a uniformly continuous h1 clone homomorphism onto $\mathbf{1}$.

Then \mathcal{C} has at least double exponential orbit growth.

Proof. By the results from [BOP], (ii) together with oligomorphicity implies that there exists $\ell \geq 1$ such that the componentwise action of \mathcal{C} on C^ℓ , which we denote by \mathcal{C}^ℓ , has a retractional witness (in the terminology of [BOP], which we avoid to fully define here, the clone $\mathbf{1}$ is an expansion of a reflection of a finite power of \mathcal{C} , which implies our formulation – see also Section 5). By Lemma 3.3, \mathcal{C}^ℓ has infinite ambiguity degree, and so it has at least double exponential orbit growth by Lemma 3.2. Hence, also \mathcal{C} has at least double exponential orbit growth. \square

Corollary 3.5. *Let \mathbb{A} be an ω -categorical model-complete core, and suppose that $\text{Pol}(\mathbb{A})$ satisfies (i) and (ii) of Theorem 3.4. Then \mathbb{A} has at least double exponential orbit growth.*

Note that this implies, in particular, Theorem 1.7. We obtain the following result in the language of clone homomorphisms, for reducts of homogeneous structures in a finite language.

Corollary 3.6. *Let \mathbb{A} be a reduct of a structure which is homogeneous in a finite relational language, and suppose \mathbb{A} is a model-complete core. Then the following are equivalent.*

- (i) *Some stabilizer of $\text{Pol}(\mathbb{A})$ has a continuous clone homomorphism to $\mathbf{1}$.*
- (ii) *$\text{Pol}(\mathbb{A})$ has a uniformly continuous h1 clone homomorphism to $\mathbf{1}$.*

Proof. The implication from (i) to (ii) is a direct consequence of the results in [BOP]. For the other direction, assume that (ii) holds. By Theorem 1.3, (i) holds if and only if $\text{Pol}(\mathbb{A})$ has no Siggers term modulo outer unary functions. If that was not the case, then Corollary 3.5 would imply that $\text{Aut}(\mathbb{A})$ has at least double exponential orbit growth, contradicting that \mathbb{A} is a reduct of a structure which is homogeneous in a finite relational language (see [Mac11]). \square

Finally, we obtain the equivalence of the two CSP conjectures.

Proof of Corollary 1.8. By [BOP], $\text{Pol}(\mathbb{A})$ has a uniformly continuous h1 clone homomorphism onto $\mathbf{1}$ if and only if $\text{Pol}(\mathbb{B})$ does, so (ii) and (iii) are equivalent. Applying Corollary 3.6 to the model-complete core \mathbb{B} , and taking into account that $\text{Aut}(\mathbb{B})$ does not have faster orbit growth than $\text{Aut}(\mathbb{A})$ (for the latter, refer to the proof of the existence of the model-complete core in Section 5), the equivalence with (i) follows. \square

3.2. The counterexample. We now prove Theorem 1.6. That is, we show that in Corollary 3.6, it would not be sufficient to only require the structure \mathbb{A} to be an ω -categorical model-complete core: the assumption of being a reduct of a homogeneous structure in finite language (or more precisely, as we can see from the proof, the assumption of less than double exponential orbit growth) is indeed needed.

Our counterexample is based on the *countable atomless Boolean algebra*, i.e., the (up to isomorphism) unique countable Boolean algebra without atoms (see e.g. [Hod97]). This Boolean algebra can be described, more explicitly, as the Boolean algebra that is freely generated by a countable set of generators. Among other interesting model-theoretical properties, it is ω -categorical and has double exponential orbit growth. In the following we occasionally view this structure as a relational structure $\mathbb{B} := (B; \wedge, \vee, \neg, 0, 1)$, where the relations are

the graphs of the fundamental operations of the Boolean algebra (although we will sometimes use the same symbols for the operations of the Boolean algebra, without danger of confusion). The following two statements about \mathbb{B} are essential for the construction of our counterexample.

Lemma 3.7. *Let $\mathbb{B} = (B; \wedge, \vee, \neg, 0, 1)$ be the countable atomless Boolean algebra. Then:*

- (i) *For every finite set $C \subseteq B$ there is a binary injective $f \in \text{Pol}(\mathbb{B})$ which stabilizes all elements of C and which is symmetric modulo outer embeddings of \mathbb{B} , i.e., the identity $e_1 \circ f(x, y) = e_2 \circ f(y, x)$ holds for some self-embeddings e_1, e_2 of \mathbb{B} .*
- (ii) *$\text{Pol}(\mathbb{B})$ has a uniformly continuous h1 clone homomorphism onto $\mathbf{1}$.*

Proof. Let $\{c_1, c_2, \dots\}$ be a countable set that freely generates \mathbb{B} . Since every element of \mathbb{B} can be expressed as a term over \mathbb{B} using finitely many generators, we can restrict ourselves in (i) to the stabilizers of sets of the form $C = \{c_1, \dots, c_n\}$. The product algebra $\mathbb{B} \times \mathbb{B}$ is also a countable atomless Boolean algebra and thus isomorphic to \mathbb{B} . Moreover, it is freely generated by the pairs $(1, 0)$ and (c_i, c_i) for all $i \geq 1$: let $(a, b) \in B \times B$, let ϕ, ψ be terms over \mathbb{B} such that $a = \phi(c_1, \dots, c_n)$ and $b = \psi(c_1, \dots, c_n)$ in \mathbb{B} . Then we can represent the pair (a, b) by

$$(a, b) = (\phi((c_1, c_1), \dots, (c_n, c_n)) \wedge (1, 0)) \vee (\psi((c_1, c_1), \dots, (c_n, c_n)) \wedge \neg(1, 0)).$$

We now define $f: \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$ to be the unique homomorphism that extends the following map between the generating sets:

$$\begin{aligned} (1, 0) &\mapsto c_{n+1}, \\ (c_i, c_i) &\mapsto c_i \text{ for all } 1 \leq i \leq n, \\ (c_i, c_i) &\mapsto c_{i+2} \text{ for all } i > n. \end{aligned}$$

By definition f is a polymorphism of \mathbb{B} that stabilizes all the elements c_i for $1 \leq i \leq n$. Furthermore, since f is induced by a bijection between the generating sets of free Boolean algebras, f is an isomorphism between $\mathbb{B} \times \mathbb{B}$ and \mathbb{B} . It satisfies the equation $f(x, y) = e \circ f(y, x)$, where e denotes the unique automorphism of \mathbb{B} that maps c_{n+1} to $\neg c_{n+1}$ and fixes all other generating elements, which concludes the proof of (i).

In order to show (ii), let F be an ultrafilter of \mathbb{B} ; we remark that some version of the axiom of choice is needed for its existence. Then for every $f \in \text{Pol}(\mathbb{B})$ exactly one of the elements $a_1 := f(1, 0, 0, \dots, 0)$, $a_2 := f(0, 1, 0, \dots, 0)$, \dots , $a_n := f(0, 0, 0, \dots, 1)$ is an element of F : this follows from the fact that, since f stabilizes 1, the disjunction $a_1 \vee \dots \vee a_n$ is equal to 1; but on the other hand $a_i \wedge a_j$ is equal to 0 whenever $i \neq j$, since f stabilizes 0. Let i_f be the unique index such that $a_{i_f} \in F$. Then $\xi(f) := \pi_{i_f}^n \in \mathbf{1}$ defines an h1 clone homomorphism from $\text{Pol}(\mathbb{B})$ to $\mathbf{1}$. Furthermore ξ is uniformly continuous, since for every $n \geq 1$ the image $\xi(f)$ of an n -ary polymorphism f only depends on the restriction of f to the finite set $\{0, 1\}^n$. \square

Note that the countable atomless Boolean algebra \mathbb{B} is not a model-complete core, since it can be homomorphically mapped to the two-element Boolean algebra. However, a slight change of language yields a model-complete core which satisfies all conditions of Theorem 1.6:

Proof of Theorem 1.6. Let \mathbb{A} be the expansion of the countable Boolean algebra \mathbb{B} by the inequality relation. Then clearly \mathbb{A} and \mathbb{B} have the same automorphism group. Using the fact

that \mathbb{A} contains the inequality relation, it can be easily verified that $\text{Aut}(\mathbb{A})$ is dense in the endomorphisms of \mathbb{A} , and so \mathbb{A} is an ω -categorical model-complete core.

By Lemma 3.7 (i), every stabilizer of $\text{Pol}(\mathbb{B})$ contains an injective binary function which is symmetric modulo outer embeddings. Since those functions are injective, they preserve in particular the inequality relation and are thus elements of $\text{Pol}(\mathbb{A})$. Therefore no stabilizer of $\text{Pol}(\mathbb{A})$ has a clone homomorphism to $\mathbf{1}$. But, by Lemma 3.7 (ii) there is a uniformly continuous h1 clone homomorphism of $\text{Pol}(\mathbb{B})$ to $\mathbf{1}$, and its restriction to $\text{Pol}(\mathbb{A})$ shows that also $\text{Pol}(\mathbb{B})$ has such a clone homomorphism. \square

3.3. The Ramsey property. We now prove Theorem 1.9, which states that also under different, Ramsey-theoretic conditions, the satisfaction of a non-trivial set of linear identities modulo outer embeddings in a polymorphism clone implies that this clone has no uniformly continuous h1 clone homomorphism to $\mathbf{1}$. Although the cases covered by this result are not congruent with the range of Conjecture 1.2, they appear in many known classifications of CSPs over homogeneous structures; in fact such CSP classifications are often based on the fact that the underlying structures can be expanded to Ramsey structures (cf. [BP11, BP16b] for numerous examples and further references).

Let \mathbb{A} be a reduct of an ordered homogeneous Ramsey structure \mathbb{D} and let $\text{Pol}(\mathbb{A})$ satisfy a non-trivial set of linear identities modulo outer embeddings of \mathbb{D} ; by homogeneity, those embeddings are elements of $\overline{\text{Aut}(\mathbb{D})}$. Then Theorem 1.9 claims that there is no uniformly continuous h1 clone homomorphism from $\text{Pol}(\mathbb{A})$ to $\mathbf{1}$. We provide two proofs, a combinatorial one applying the Ramsey property directly, and a more algebraic one using dynamical systems.

First proof of Theorem 1.9. Let $\text{Pol}(\mathbb{A})$ satisfy the non-trivial set of identities

$$u^i \circ s^i(y_1^i, \dots, y_m^i) = v^i \circ t^i(z_1^i, \dots, z_m^i),$$

where $u^i, v^i \in \overline{\text{Aut}(\mathbb{D})}$, $s^i, t^i \in \text{Pol}(\mathbb{A})$, and y_j^i, z_j^i are not necessarily distinct variables, for $1 \leq i \leq k$ and $1 \leq j \leq m$. The finiteness of this set follows from the compactness theorem of first-order logic, and we can assume $s^1, \dots, s^k, t^1, \dots, t^k$ to have equal arity m by adding dummy variables. Moreover assume, for technical reasons, that every right side of an identity also appears as a left side, simply by repeating identities. For contradiction, let us assume that there is a uniformly continuous h1 clone homomorphism $\xi: \text{Pol}(\mathbb{A}) \rightarrow \mathbf{1}$.

By the uniform continuity of ξ there is a finite $F \subseteq A$ such that whenever two functions $f, g \in \text{Pol}(\mathbb{A})$ of arity m agree on F , then $\xi(f) = \xi(g)$. We are going to color the copies of the structures $\mathbb{C}^1, \dots, \mathbb{C}^k$ induced in \mathbb{D} by $s^1[F^m], \dots, s^k[F^m]$.

By the homogeneity of \mathbb{D} all such copies have domains of the form $\alpha[s^i[F^m]]$, where $\alpha \in \text{Aut}(\mathbb{D})$. Since \mathbb{D} is totally ordered, every other $\beta \in \text{Aut}(\mathbb{D})$ that maps $s^i[F^m]$ to $\alpha[s^i[F^m]]$ has to coincide with α on $s^i[F^m]$. Hence, since $\xi(\alpha \circ s^i)$ only depends on the restriction of $\alpha \circ s^i$ to F^m , the coloring χ^i on the copies of \mathbb{C}^i which sends every copy induced by $\alpha[s^i[F^m]]$ to $\xi(\alpha \circ s^i)$ is well defined.

Now set \mathbb{S} to be structure induced by the union of all sets $s^i[F^m]$ and $u^i[s^i[F^m]]$, where $1 \leq i \leq k$. By the Ramsey property, there is an isomorphic copy \mathbb{S}' of \mathbb{S} in \mathbb{D} on which the colorings χ^i are monochromatic. This implies that for any $\beta \in \text{Aut}(\mathbb{D})$ that maps \mathbb{S} to \mathbb{S}' we have $\xi(\beta \circ u^i \circ s^i) = \xi(\beta \circ s^i)$ for all $1 \leq i \leq k$. Hence, because ξ preserves the linear identities above, $\xi(\beta \circ s^i(y_1^i, \dots, y_m^i)) = \xi(\beta \circ t^i(z_1^i, \dots, z_m^i))$, contradicting the fact that the system of identities of the form $\tilde{s}^i(y_1^i, \dots, y_m^i) = \tilde{t}^i(z_1^i, \dots, z_m^i)$ is not satisfiable in $\mathbf{1}$. \square

Second proof of Theorem 1.9. We will use the fact due to [KPT05] that $\text{Aut}(\mathbb{D})$ is, as the automorphism group of an ordered Ramsey structure, *extremely amenable*: whenever it acts continuously on a compact Hausdorff space, then this action has a fixed point.

Fix $m \geq 1$, and let S_m be the set of all mappings from the m -ary functions in $\text{Pol}(\mathbb{A})$ to the m -ary functions in $\mathbf{1}$. Bearing the product topology, S_m is a compact Hausdorff space. We define an action of $\text{Aut}(\mathbb{D})$ on S_m by setting, for $\alpha \in \text{Aut}(\mathbb{D})$ and $\xi \in S_m$, the mapping $(\alpha \cdot \xi) \in S_m$ to be given by

$$(\alpha \cdot \xi)(f) := \xi(\alpha^{-1} \circ f) \text{ for all } m\text{-ary } f \in \text{Pol}(\mathbb{A}).$$

For contradiction, suppose that there is a uniformly continuous h1 clone homomorphism from $\text{Pol}(\mathbb{A})$ to $\mathbf{1}$, and let $\xi \in S_m$ be its restriction to m -ary functions, where $m \geq 1$ is fixed. Consider the restriction of the above action of $\text{Aut}(\mathbb{D})$ to the closure of the orbit of ξ in S_m , i.e., let $\text{Aut}(\mathbb{D})$ act on

$$C := \overline{\{\alpha \cdot \xi \mid \alpha \in \text{Aut}(\mathbb{D})\}}.$$

Clearly, C is compact. Moreover, this restriction of the action to C is continuous: to illustrate this, let us first observe that there exists a finite set $F \subseteq A^m$ such that for all $\psi \in C$ and all m -ary $f, f' \in \text{Pol}(\mathbb{A})$ we have that $f|_F = f'|_F$ implies $\psi(f) = \psi(f')$. Now consider a basic open neighborhood

$$O_f(\psi) = \{\psi' \in C \mid \psi'(f) = \psi(f)\}$$

of some $\psi \in C$, where the m -ary $f \in \text{Pol}(\mathbb{A})$ is fixed. Then by our remark above, the set

$$\{(\alpha, \psi') \in \text{Aut}(\mathbb{D}) \times C \mid \alpha \text{ stabilizes } f[F], \text{ and } \psi'(f) = \psi(f)\}$$

is a basic open neighborhood of (id, ψ) that is mapped into $O_f(\psi)$ under the action.

Since $\text{Aut}(\mathbb{D})$ is extremely amenable, there is a fixed point ξ' of its action on C , i.e., $(\alpha \cdot \xi') = \xi'$ for all $\alpha \in \text{Aut}(\mathbb{D})$. This means that ξ' preserves composition with elements of $\text{Aut}(\mathbb{D})$ from the outside, and by continuity even with elements of $\overline{\text{Aut}(\mathbb{D})}$, i.e., with self-embeddings of \mathbb{D} . Moreover, ξ' preserves linear identities, since any mapping $\alpha \cdot \xi$ does, and so does any mapping in the closure of the functions of the latter form.

It follows that $\text{Pol}(\mathbb{A})$ cannot satisfy any finite non-trivial set of identities which are linear modulo embeddings of \mathbb{D} from the outside, as otherwise they would be satisfied in $\mathbf{1}$ by virtue of ξ' , if we choose m larger than all arities of the functions in that set. \square

We would like to remark that the atomless Boolean algebra \mathbb{B} that was used to provide the counterexample of Theorem 1.6 is the reduct of a homogeneous Ramsey structure, namely of its expansion by a linear order which extends the natural partial order on \mathbb{B} (see for instance [KPT05]). We proved in Lemma 3.7 that there are polymorphisms of \mathbb{B} satisfying the non-trivial equation $f(x, y) = e \circ f(y, x)$. However this non-trivial equation does not satisfy the condition of Theorem 1.9, since the embedding e does not preserve any linear order on the domain of \mathbb{B} .

4. LINEARIZATION OF NON-TRIVIAL IDENTITIES

We are going to show that, under stronger conditions than the existence of a Siggers term modulo outer embeddings, we can derive the satisfaction of non-trivial linear identities in polymorphism clones. In Section 4.1 we prove a strengthening of Theorem 1.10. A proof of Theorem 1.11 is given in Section 4.2. Finally, in Section 4.3 we show how to apply these results to the polymorphism clones of all reducts of equality, the rational order, the random

graph and the random partial order, for which complete complexity classifications of the corresponding CSPs have been obtained [BK08, BK09, BP15a, KP].

4.1. Totally symmetric polymorphisms modulo embeddings. The mentioned strengthening of Theorem 1.10, Proposition 4.2, uses a weaker notion of total symmetry.

Definition 4.1. Let $f(x_1, \dots, x_k)$ be a k -ary operation on a set D . We define the *nu-minors* of f as the binary functions $h_i^f(x, y) := f(x, \dots, x, y, x, \dots, x)$, where $1 \leq i \leq k$ and the only y is located on the i -th coordinate. When \mathbb{D} is a relational structure on D , we say that the nu-minors of f are *totally symmetric modulo outer embeddings* of \mathbb{D} if for all permutations ρ of $\{1, \dots, k\}$ there are embeddings e_ρ, e'_ρ of \mathbb{D} such that

$$e_\rho \circ h_i^f(x, y) = e'_\rho \circ h_{\rho(i)}^f(x, y) \text{ for all } 1 \leq i \leq k.$$

Clearly, whenever f is totally symmetric modulo outer embeddings of \mathbb{D} , then also its nu-minors are totally symmetric modulo outer embeddings of \mathbb{D} . On the other hand, we remark that if f is a *weak near unanimity* function modulo outer embeddings of \mathbb{D} , i.e., satisfies the identities

$$e_1 \circ h_1^f(x, y) = \dots = e_k \circ h_k^f(x, y)$$

for embeddings e_1, \dots, e_k of \mathbb{D} , then this does not imply in an obvious way that its nu-minors are symmetric modulo outer embeddings. We will show the following.

Proposition 4.2. *Let \mathbb{A} be a reduct of a finitely bounded homogeneous structure \mathbb{D} whose age is given by a finite set of forbidden substructures all of which have size at most $k \geq 3$.*

If $\text{Pol}(\mathbb{A})$ contains a k -ary function whose nu-minors are totally symmetric modulo outer embeddings of \mathbb{D} , then $\text{Pol}(\mathbb{A})$ does not have an h1 clone homomorphism to $\mathbf{1}$.

The proof of Proposition 4.2 is based on the following easy observation, which relies on the pigeonhole principle.

Lemma 4.3. *Let \mathcal{C} be a function clone, let $k \geq 2$, and assume that there are binary functions $g_i \in \mathcal{C}$ for all $i \in \{1, \dots, 2k-1\}$ such that for every injective $\psi: \{1, \dots, k\} \rightarrow \{1, \dots, 2k-1\}$ there exists $f_\psi(x_1, \dots, x_k) \in \mathcal{C}$ whose nu-minors equal the functions $g_{\psi(1)}, \dots, g_{\psi(k)}$. Then there is no h1 clone homomorphism from \mathcal{C} to $\mathbf{1}$.*

Proof. If there was an h1 clone homomorphism from \mathcal{C} onto $\mathbf{1}$, then by the pigeonhole-principle there would be $\psi: \{1, \dots, k\} \rightarrow \{1, \dots, 2k-1\}$ such that $g_{\psi(1)}, \dots, g_{\psi(k)}$ all are sent to the same projection. But this contradicts the fact that $g_{\psi(1)}, \dots, g_{\psi(k)}$ are the nu-minors of f_ψ . \square

It is further enough to find polymorphisms that satisfy the equations in Lemma 4.3 locally, by a simple compactness argument which yields the following lemma.

Lemma 4.4 (Lemma 3 in [BP16b]). *Let \mathbb{D} be an ω -categorical structure. Let J be an index set, and for every $j \in J$, let f_j and g_j be functions on D of the same arity $m \geq 1$ such that for every finite $F \subseteq A^m$ there is $\alpha_j \in \text{Aut}(\mathbb{D})$ with $\alpha_j \circ f_j|_F = g_j|_F$. Then there are $(e_j)_{j \in J}, e \in \overline{\text{Aut}(\mathbb{D})}$ such that $e_j \circ f_j = e \circ g_j$ for all $j \in J$. Moreover, if for $j_1, j_2 \in J$ we have $\alpha_{j_1} = \alpha_{j_2}$ for every finite set F , then $e_{j_1} = e_{j_2}$.*

We are going to construct the functions g_i needed for Lemma 4.3 as suitable compositions of the nu-minors of f with embeddings of \mathbb{D} .

Lemma 4.5. *Let \mathbb{A} , \mathbb{D} , and $f(x_1, \dots, x_k)$ be as in Proposition 4.2, and let $F \subseteq A$ be finite. Then there are binary $g_1, \dots, g_{2k-1} \in \text{Pol}(\mathbb{A})$ such that for every injective $\psi: \{1, \dots, k\} \rightarrow \{1, \dots, 2k-1\}$ there exists $\alpha_\psi \in \text{Aut}(\mathbb{D})$ such that $g_{\psi(i)} \upharpoonright_{F^2} = \alpha_\psi \circ h_i^f \upharpoonright_{F^2}$ for all $1 \leq i \leq k$.*

Proof. Whenever $\psi: \{1, \dots, k\} \rightarrow \{1, \dots, 2k-1\}$ is injective, we define a mapping φ_ψ

$$\begin{aligned} F^2 \times \psi[\{1, \dots, k\}] &\rightarrow D \\ (a, b, i) &\mapsto h_{\psi^{-1}(i)}^f(a, b). \end{aligned}$$

Writing \sim_ψ for the kernel of φ_ψ , we then naturally obtain a structure \mathbb{X}_ψ in the language of \mathbb{D} on the set $(F^2 \times \psi[\{1, \dots, k\}]) / \sim_\psi$ of kernel classes of ψ , in which we choose the relations to be so that the mapping from \mathbb{X}_ψ to \mathbb{D} induced by φ_ψ is an embedding.

The main point of our construction is the observation that because the nu-minors of f are totally symmetric modulo outer embeddings of \mathbb{D} , any two structures $\mathbb{X}_{\psi_1}, \mathbb{X}_{\psi_2}$ are isomorphic via the mapping that sends any kernel class $[(a, b, i)]_{\sim_{\psi_1}}$ to $[(a, b, \psi_2 \circ \psi_1^{-1}(i))]_{\sim_{\psi_2}}$. For example, to see that this mapping is well-defined, note that by definition $(a, b, i) \sim_{\psi_1} (c, d, j)$ if and only if $h_{\psi_1^{-1}(i)}^f(a, b) = h_{\psi_1^{-1}(j)}^f(c, d)$; but this is the case, by definition, if and only if $(a, b, \psi_2 \circ \psi_1^{-1}(i)) \sim_{\psi_2} (c, d, \psi_2 \circ \psi_1^{-1}(j))$. Similarly, one checks that the mapping is an isomorphism.

We define a binary relation \sim on $F^2 \times \{1, \dots, 2k-1\}$ by setting $(a, b, i) \sim (c, d, j)$ if and only if there is a ψ such that $(a, b, i) \sim_\psi (c, d, j)$, and claim that it is transitive, and thus an equivalence relation. To see transitivity, note that by the total symmetry of nu-minors, $(a, b, i) \sim (c, d, j)$ is equivalent to the statement that for every ψ containing i and j in its image $(a, b, i) \sim_\psi (c, d, j)$ holds. Now let $(a, b, i), (c, d, j), (u, v, m) \in F^2 \times \{1, \dots, 2k-1\}$ with $(a, b, i) \sim (c, d, j)$ and $(c, d, j) \sim (u, v, m)$. Since $k \geq 3$, there is an injection ψ such that $(a, b, i) \sim_\psi (c, d, j) \sim_\psi (u, v, m)$, and hence $(a, b, i) \sim_\psi (u, v, m)$.

Since the relations of the structures \mathbb{X}_ψ agree on their intersections, we obtain a structure \mathbb{X} on the equivalence classes of \sim , defined as the “union” of the structures \mathbb{X}_ψ . This structure \mathbb{X} does not contain any forbidden substructures of \mathbb{D} , since any k -element substructure of \mathbb{X} is already contained in some structure \mathbb{X}_ψ , which in turn embeds into \mathbb{D} . Hence, \mathbb{X} embeds into \mathbb{D} via an embedding φ . For $1 \leq i \leq 2k-1$ and $(a, b) \in F^2$, we now set $g_i(a, b) := \varphi([(a, b, i)]_\sim)$.

Given $\psi: \{1, \dots, k\} \rightarrow \{1, \dots, 2k-1\}$ as in the lemma, it is clear from the definition of \mathbb{X}_ψ that the tuples $(g_{\psi(i)}(a, b) \mid 1 \leq i \leq k, (a, b) \in F^2)$ and $(h_i^f(a, b) \mid 1 \leq i \leq k, (a, b) \in F^2)$ satisfy the same relations in \mathbb{D} . By the homogeneity of \mathbb{D} , the latter can be sent to the first via an automorphism α_ψ of \mathbb{D} , which is what we had to show. \square

We have now all the tools ready to prove Proposition 4.2.

Proof of Proposition 4.2. Let $f(x_1, \dots, x_k) \in \text{Pol}(\mathbb{A})$ have totally symmetric nu-minors. In Lemma 4.5 we showed that for every finite $F \subseteq A$ we find functions $g_i \in \text{Pol}(\mathbb{A})$, for $1 \leq i \leq 2k-1$, such that for every injective mapping $\psi: \{1, \dots, k\} \rightarrow \{1, \dots, 2k-1\}$ there is an automorphism $\alpha_\psi \in \text{Aut}(\mathbb{D})$ with $g_{\psi(i)} \upharpoonright_{F^2} = \alpha_\psi \circ h_i^f \upharpoonright_{F^2}$ for all $1 \leq i \leq k$. By Lemma 4.4 we obtain embeddings $e, e_\psi \in \overline{\text{Aut}(\mathbb{D})}$ such that $e \circ g_{\psi(i)} = e_\psi \circ h_i^f$ for all $1 \leq i \leq k$. Then the functions $f_\psi := e_\psi \circ f$ and their nu-minors $e \circ g_{\psi(1)}, \dots, e \circ g_{\psi(k)}$ satisfy the conditions of Lemma 4.3, which concludes the proof. \square

Observing that the assumption $k \geq 3$ was only needed to “amalgamate” the kernels in the proof of Lemma 4.5, we obtain the following variant of Proposition 4.2 in which we trade the condition $k \geq 3$ for injectivity.

Corollary 4.6. *Let \mathbb{A} be a reduct of a finitely bounded homogeneous structure \mathbb{D} whose age is given by a finite set of forbidden substructures all of which have size at most $k \geq 2$.*

If $\text{Pol}(\mathbb{A})$ contains a k -ary function which is injective and whose nu-minors are totally symmetric modulo outer embeddings of \mathbb{D} , then $\text{Pol}(\mathbb{A})$ does not have an h1 clone homomorphism to $\mathbf{1}$.

4.2. Near unanimity polymorphisms modulo embeddings. We are now going to prove Theorem 1.11. The proof is, similarly to the proof of Proposition 4.2, going to use Lemma 4.3. In Section 4 we constructed the binary functions g_i needed for the proof from totally symmetric (modulo outer embeddings of \mathbb{D}) nu-minors, and used the fact that these minors uniformly show a certain regular behaviour. Here, we are going to construct the functions g_i as nu-minors of a function which we are going to construct from f , and these nu-minors are going to uniformly show regular behaviour in the following sense. Call a set $\{g_1, \dots, g_n\}$ of binary operations on D *uniformly 1-dominated* if for every relation R of \mathbb{D} , including equality, we have the following equivalence for all $i_1, \dots, i_t \in \{1, \dots, n\}$ and all $a_1, \dots, a_t, b_1, \dots, b_t \in D$, where t is the arity of R :

$$(g_{i_1}(a_1, b_1), \dots, g_{i_t}(a_t, b_t)) \in R \leftrightarrow (a_1, \dots, a_t) \in R.$$

We remark that this notion is inspired by a similar concept for a single function in [BP15a, BP14].

In the following, for a function $f: D^k \rightarrow D$, we set $f^1 := f$, and define recursively $f^{n+1} := f * f^n$ for every $n \geq 1$ (where $*$ bears the same meaning as in Section 3).

Lemma 4.7. *Let \mathbb{D} be a homogeneous structure in a finite relational language. Suppose that f is a strong polymorphism of \mathbb{D} that is nu modulo outer embeddings of \mathbb{D} . If $n \geq 2$ exceeds the maximal arity of the relations of \mathbb{D} , then the set of nu-minors of f^n is uniformly 1-dominated.*

Proof. Let $k \geq 1$ be the arity of f ; so f^n is k^n -ary. As in Definition 4.1, we denote the nu-minors of f^n by $h_i^{f^n}(x, y)$, for $1 \leq i \leq k^n$. Let R be a t -ary relation of \mathbb{D} , and let $I := \{i_1, \dots, i_t\} \subseteq k^n$ and $a_1, \dots, a_t, b_1, \dots, b_t \in D$. We have to show

$$(h_{i_1}^{f^n}(a_1, b_1), \dots, h_{i_t}^{f^n}(a_t, b_t)) \in R \leftrightarrow (a_1, \dots, a_t) \in R.$$

Note that the tuple $(h_{i_1}^{f^n}(a_1, b_1), \dots, h_{i_t}^{f^n}(a_t, b_t))$ can also be written as $f^n(\bar{c}_1, \dots, \bar{c}_{k^n})$, where each \bar{c}_i is a t -tuple (c_i^1, \dots, c_i^t) and each $c_i^j \in \{a_j, b_j\}$. The value of c_i^j depends on $h_{i_j}^{f^n}$, but for fixed $1 \leq j \leq t$ all but one c_i^j equal a_j , and the remaining one b_j . Consequently all but t of the tuples \bar{c}_i equal the tuple (a_1, \dots, a_t) .

By the above, we can prove the implication $(a_1, \dots, a_t) \in R \rightarrow f^n(\bar{c}_1, \dots, \bar{c}_{k^n}) \in R$ by showing the following claim for every $n \geq 1$: whenever f^n is applied to t -tuples $\bar{d}_1, \dots, \bar{d}_{k^n}$, all of which with the exception of less than n many are equal and in R , then $f^n(\bar{d}_1, \dots, \bar{d}_{k^n}) \in R$. We proceed by induction. The case $n = 1$ is clear since f preserves R . In the induction step, we consider f^n for $n \geq 2$. Noting that the k maximal proper subterms of $f^n = f * f^{n-1}$ are of the form f^{n-1} (with the right choice variables), and that f^n is f applied to these k subterms, we distinguish two cases: if all “exceptional” tuples \bar{d}_i appear in one of these subterms, then the claim follows immediately from the fact that f is nu modulo embeddings. Otherwise,

each of these subterms contains less than $n - 1$ exceptions, and we can apply the induction hypothesis to see that each of them yields an element of R . Hence, $f^n(\bar{d}_1, \dots, \bar{d}_{k^n}) \in R$ since f preserves R .

For the other implication, namely $(a_1, \dots, a_t) \notin R \rightarrow f^n(\bar{c}_1, \dots, \bar{c}_{k^n}) \notin R$, we can use the same argument, noting that f preserves the negation of R . \square

Proof of Theorem 1.11. Let $f: D^k \rightarrow D$ be the strong polymorphism of \mathbb{D} which is nu modulo outer embeddings of \mathbb{D} . We use the notation of Lemma 4.7, and fix $n \geq 2$ so that it exceeds the maximal arity of the relations of \mathbb{D} . Then, by Lemma 4.7, the set of nu-minors of f^n is uniformly 1-dominated, and so is the set of nu-minors of f^{n+1} .

We are going to apply Lemma 4.3. The binary functions g_i in that lemma will be the first $2k^n - 1$ nu-minors of f^{n+1} , composed with appropriate embeddings of \mathbb{D} ; the functions f_ψ will be obtained by composing f^n with appropriate embeddings of \mathbb{D} . In particular, the arity of the f_ψ will be k^n .

For the precise construction, we claim that for every finite $F \subseteq D^2$ and every injective $\psi: \{1, \dots, k^n\} \rightarrow \{1, \dots, 2k^n - 1\}$, there is a function $\alpha_\psi \in \text{Aut}(\mathbb{D})$ such that

$$h_{\psi(j)}^{f^{n+1}} \upharpoonright_F = \alpha_\psi \circ h_j^{f^n} \upharpoonright_F$$

holds for all $1 \leq j \leq k^n$. Since the nu-minors of f^n and f^{n+1} are 1-dominated, the map sending every $h_j^{f^n}(a, b)$ to $h_{\psi(j)}^{f^{n+1}}(a, b)$, for every $(a, b) \in F$, is well-defined and preserves all relations of \mathbb{D} and their negations. Hence, the existence of the automorphism α_ψ follows from the homogeneity of \mathbb{D} .

By Lemma 4.4 we obtain embeddings $e, e_\psi \in \overline{\text{Aut}(\mathbb{D})}$ such that $e \circ h_{\psi(j)}^{f^{n+1}} = e_\psi \circ h_j^{f^n}$. Now the functions $g_j(x, y) := e \circ h_j^{f^{n+1}}$ and $f_\psi = e_\psi \circ f^n$ satisfy the conditions of Lemma 4.3. \square

4.3. Examples of linearization. We are now going to prove Theorem 1.12. That is, we are going to show that for any reduct \mathbb{A} of equality, the order of the rationals, the random partial order, or the random graph, $\text{Pol}(\mathbb{A})$ has no uniformly continuous h1 clone homomorphism to $\mathbf{1}$ if and only if it satisfies a non-trivial set of linear identities. To this end, we are going to analyse the linear identities modulo embeddings obtained in the corresponding CSP classifications [BK08, BK09, KP, BP15a]. In most of the cases, Theorem 1.10 and Theorem 1.11 provide us directly with the desired linear identities, but we do have to consider some cases separately. We present the proof for $(\mathbb{N}; =)$ in Proposition 4.9, for the order of the rationals in Proposition 4.11, for the random partial order in Proposition 4.13, and for the random graph in Proposition 4.16.

4.3.1. Reducts of equality. For the reducts of $(\mathbb{N}; =)$, the CSP classification in [BK08] shows the following.

Theorem 4.8. *Let \mathbb{A} be a reduct of $(\mathbb{N}; =)$. Then either*

- (1) *$\text{Pol}(\mathbb{A})$ contains a constant or a binary injective function, or*
- (2) *there is a continuous clone homomorphism from $\text{Pol}(\mathbb{A})$ to $\mathbf{1}$.*

If \mathbb{A} has a finite relational language, then $\text{CSP}(\mathbb{A})$ is tractable in the first case, and NP-complete in the second case.

Theorem 1.10 then yields Theorem 1.12 for such reducts.

Proposition 4.9. *Theorem 1.12 holds for reducts of $(\mathbb{N}; =)$.*

Proof. If a reduct \mathbb{A} has a constant polymorphism, then it has a binary such polymorphism c , which clearly satisfies the non-trivial linear identity $c(x, y) = c(y, x)$. If $\text{Pol}(\mathbb{A})$ contains a binary injection $f(x, y)$, then $f(x, f(y, z))$ is an injective ternary polymorphism of \mathbb{A} which satisfies the conditions of Theorem 1.10. \square

4.3.2. *Reducts of the order of the rational numbers.* For the reducts of $(\mathbb{Q}; \leq)$ the CSP classification in [BK09] shows the following, using the notation of [BK09, Bod12].

Theorem 4.10. *Let \mathbb{A} be a reduct of $(\mathbb{Q}; \leq)$. Then either*

- (1) *$\text{Pol}(\mathbb{A})$ contains one of the operations $\min, \text{mi}, \text{mx}, \text{ll}$, their duals, or a constant,*
- (2) *there is a continuous clone homomorphism from $\text{Pol}(\mathbb{A})$ to $\mathbf{1}$.*

If \mathbb{A} has a finite relational language, then $\text{CSP}(\mathbb{A})$ is tractable in the first case, and NP-complete in the second case.

Proposition 4.11. *Theorem 1.12 holds for reducts of the order of the rationals $(\mathbb{Q}; \leq)$.*

Proof. It suffices to show for a reduct \mathbb{A} that if $\text{Pol}(\mathbb{A})$ contains one of the operations in Theorem 4.10 (1), then it satisfies non-trivial linear identities. This is clear if $\text{Pol}(\mathbb{A})$ contains a constant operation. The operations mx and \min satisfy the non-trivial linear identities $\text{mx}(x, y) = \text{mx}(y, x)$ and $\min(x, y) = \min(y, x)$, respectively.

For the case when $\text{mi} \in \text{Pol}(\mathbb{A})$ we are going to sketch a proof using Lemma 4.3. Let $\beta, \alpha_i, \gamma_i$ be self-embeddings of $(\mathbb{Q}; \leq)$ for $i, j \in \{1, \dots, 5\}$ such that

$$\begin{aligned} \beta(x) &< \gamma_1(x) < \gamma_2(x) < \dots < \gamma_5(x) \\ &< \alpha_1(x) < \alpha_2(x) < \dots < \alpha_5(x) < \beta(x + \epsilon) \end{aligned}$$

for every $x \in \mathbb{Q}$ and every $0 < \epsilon \in \mathbb{Q}$. Then for $1 \leq i \leq 5$, the functions

$$\text{mi}_i(x, y) := \begin{cases} \alpha_i(x) & \text{if } x < y \\ \beta(x) & \text{if } x = y \\ \gamma_i(y) & \text{if } x > y \end{cases}$$

can be written as a composition of mi with embeddings of $(\mathbb{Q}; \leq)$, thus they are polymorphisms of \mathbb{A} . Following the proof of Proposition 10.5.17 in [Bod12], for each injection $\psi: \{1, 2, 3\} \rightarrow \{1, \dots, 5\}$ we can construct $f_\psi \in \text{Pol}(\mathbb{A})$ such that there is an embedding $e \in \overline{\text{Aut}(\mathbb{Q}; \leq)}$ with

$$\begin{aligned} f_\psi(y, x, x) &= e \circ \text{mi}_{\psi(1)}(y, x) \\ f_\psi(x, y, x) &= e \circ \text{mi}_{\psi(2)}(y, x) \\ f_\psi(x, x, y) &= e \circ \text{mi}_{\psi(3)}(y, x); \end{aligned}$$

hence we found functions satisfying the conditions of Lemma 4.3.

We are left with the case where $\text{Pol}(\mathbb{A})$ contains the binary function ll . Then by Proposition 10.4.10 in [Bod12], $\text{Pol}(\mathbb{A})$ also contains

$$\begin{aligned} f(x, y, z) &:= \text{lex}'(\min(x, y, z), \\ &\quad \max(\min(x, y), \min(x, z), \min(y, z)), \\ &\quad x, y, z), \end{aligned}$$

where lex' is a 5-ary operation that embeds the lexicographical order on $(\mathbb{Q}; \leq)^5$ into the order $(\mathbb{Q}; \leq)$. Analogously to the existence of f , one can show that $\text{Pol}(\mathbb{A})$ contains the operation

$$\begin{aligned} g(x, y, z, t, u) := & \text{lex}(\min(x, y, z, t, u), \\ & \max(\min(x, y), \min(x, z), \min(x, t), \min(x, u), \\ & \min(y, z), \min(y, t), \min(y, u), \min(z, t), \min(z, u)), \\ & x, y, z, t, u), \end{aligned}$$

where lex embeds the lexicographic order on $(\mathbb{Q}; \leq)^7$ into $(\mathbb{Q}; \leq)$. Let h_1^g, \dots, h_5^g be the minors of g . For every finite $F \subseteq \mathbb{Q}$ and every injective $\psi: \{1, 2, 3\} \rightarrow \{1, \dots, 5\}$, it can be easily verified that there is $\alpha_\psi \in \text{Aut}(\mathbb{Q}; \leq)$ such that on F the identities $\alpha_\psi \circ f(y, x, x) = h_{\psi(1)}^g(x, y)$, $\alpha_\psi \circ f(x, y, x) = h_{\psi(2)}^g(x, y)$, and $\alpha_\psi \circ f(x, x, y) = h_{\psi(3)}^g(x, y)$ hold. Lemma 4.4 then yields a set of functions that satisfies the non-trivial linear identities of Lemma 4.3. \square

4.3.3. Reducts of the random partial order. In the complexity classification of CSPs for reducts of the random partial order, which we denote by \mathbb{P} , the following dichotomy has been shown [KP]; we use the definitions from that article.

Theorem 4.12. *Let \mathbb{A} be a reduct of \mathbb{P} . Then one of the following applies.*

- (1) \mathbb{A} is homomorphically equivalent to a reduct of $(\mathbb{Q}; \leq)$.
- (2) $\text{Pol}(\mathbb{A})$ contains the binary operation $e_<$ or e_\leq .
- (3) There is a continuous clone homomorphism from $\text{Pol}(\mathbb{A})$ to $\mathbf{1}$.

If \mathbb{A} has a finite relational language, then the third case implies that $\text{CSP}(\mathbb{A})$ is NP-complete, and the second case implies tractability of the CSP.

Proposition 4.13. *Theorem 1.12 holds for reducts of the random partial order \mathbb{P} .*

Proof. If a reduct \mathbb{A} is homomorphically equivalent to a reduct of $(\mathbb{Q}; \leq)$, then the statement follows from Proposition 4.11 and the fact that homomorphic equivalence preserves linear identities (Theorem 1.5 in [BOP]). If item (2) applies, then note that the mappings $(x, y, z) \mapsto e_<(e_<(x, y), z)$ and $(x, y, z) \mapsto e_\leq(e_\leq(x, y), z)$ are totally symmetric modulo outer embeddings. Since \mathbb{P} can be described by forbidden substructures of size 3, the satisfaction of non-trivial linear identities in $\text{Pol}(\mathbb{A})$ follows from Theorem 1.10. \square

4.3.4. Reducts of random graph. For the random graph $G = (V; E)$, the following dichotomy has been shown [BP15a, BP11].

Theorem 4.14. *Let \mathbb{A} be a reduct of G . Then one of the following holds:*

- (1) $\text{Pol}(\mathbb{A})$ contains a constant operation.
- (2) $\text{Pol}(\mathbb{A})$ contains an (at most ternary) injective weak near unanimity function $f(x_1, \dots, x_k)$ modulo outer embeddings of G , i.e., f satisfies identities of the form

$$\begin{aligned} e_1 \circ f(y, x, \dots, x) &= e_2 \circ f(x, y, x, \dots, x) = \dots \\ &= e_k \circ f(x, \dots, x, y), \end{aligned}$$

with $e_1, \dots, e_k \in \overline{\text{Aut}(G)}$.

- (3) $\text{Pol}(\mathbb{A})$ has a continuous homomorphism to $\mathbf{1}$.

If \mathbb{A} has a finite relational language, then (3) implies that $\text{CSP}(\mathbb{A})$ is NP-complete, and (1) and (2) imply tractability of the CSP.

In fact, [BP15a] provides a list of concrete weak near unanimity functions modulo outer embeddings that can appear, and Theorems 1.11 and 1.10 directly apply to a subset of those functions. To obtain non-trivial linear identities for all cases, and moreover simultaneously so, we are going to use the following variant of Lemma 4.3.

Lemma 4.15. *Let \mathcal{C} be a function clone, and suppose there are binary $g_{i,j} \in \mathcal{C}$ for $i, j \in \{1, \dots, k\}$ such that*

- (1) *for every fixed $j \in \{1, \dots, k\}$ there is a function $f_j(x_1, \dots, x_k) \in \mathcal{C}$ whose nu-minors equal $g_{1,j}, \dots, g_{k,j}$, and*
- (2) *for every $\psi: \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ there is a function $f_\psi(x_1, \dots, x_k) \in \mathcal{C}$ whose nu-minors equal $g_{\psi(1),1}, \dots, g_{\psi(k),k}$.*

Then there is no h1 clone homomorphism from \mathcal{C} to $\mathbf{1}$.

Proof. Suppose to the contrary that there exists an h1 clone homomorphism from \mathcal{C} to $\mathbf{1}$. If for a fixed $j \in \{1, \dots, k\}$ the functions $g_{1,j}, \dots, g_{k,j}$ are mapped to the same projection in $\mathbf{1}$, then this contradicts the fact that they are the nu-minors of $f_j(x_1, \dots, x_k)$. Thus, for every $j \in \{1, \dots, k\}$ there exists $\psi(j) \in \{1, \dots, k\}$ such that $g_{\psi(j),j}$ is mapped to the first projection. But then $g_{\psi(1),1}, \dots, g_{\psi(k),k}$, the nu-minors of f_ψ , are all sent to the same projection, a contradiction. \square

Proposition 4.16. *Theorem 1.12 holds for reducts of the random graph $G = (V; E)$.*

Proof. If $\text{Pol}(\mathbb{A})$ contains a constant operation, then the linear identity $c(x, y) = c(y, x)$ holds for some constant binary $c \in \text{Pol}(\mathbb{A})$. So we only have to study the case where $f(x_1, \dots, x_k)$ is injective and weak nu modulo outer embeddings of G .

As in Definition 4.1, denote the nu-minors of f by h_1^f, \dots, h_k^f ; we are going to construct the functions $g_{i,j}$, f_j , and f_ψ required in Lemma 4.15 from these nu-minors. By Lemma 4.4 we only have to prove for every finite $F \subseteq V$ that there are functions $g_{i,j}$, f_j , and f_ψ that satisfy the identities in Lemma 4.15 on F . To this end, we construct a graph H with vertices (i, j, x, y) , where $i, j \in \{1, \dots, k\}$ and $x, y \in F$, and in which two vertices (i_1, j_1, x_1, y_1) and (i_2, j_2, x_2, y_2) are adjacent if and only if

- $j_1 = j_2$ and $(h_{i_1}^f(x_1, y_1), h_{i_2}^f(x_2, y_2)) \in E$, or
- $j_1 \neq j_2$ and $(h_{j_1}^f(x_1, y_1), h_{j_2}^f(x_2, y_2)) \in E$.

By the universality of the random graph we can regard H as a subgraph of G . By our construction, for every $j \in \{1, \dots, k\}$ there exists $\alpha_j \in \text{Aut}(G)$ with $\alpha_j \circ h_i^f(x, y) = (i, j, x, y)$ for all $i \in \{1, \dots, k\}$ and all $x, y \in F$. Similarly for every $\psi: \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ there exists $\alpha_\psi \in \text{Aut}(G)$ such that $\alpha_\psi \circ h_i^f(x, y) = (\psi(i), i, x, y)$ for all $i \in \{1, \dots, k\}$ and all $x, y \in F$. It is easy to verify that then $g_{i,j} := \alpha_j \circ h_i^f$, $f_j := \alpha_j \circ f$ and $f_\psi := \alpha_\psi \circ f$ satisfy the equations in Lemma 4.15 on F , and we are done. \square

5. MODEL-COMPLETE CORES: A NEW PROOF

We are going to give a new and short proof of Theorem 1.1 in the language of monoids. As mentioned in Section 1.7 of the introduction, our proof will work for *weakly oligomorphic* structures, a generalization of ω -categorical structures [PP16]. Those are best defined via their endomorphism monoid.

Definition 5.1. A transformation monoid \mathcal{M} on a countable set M is called *weakly oligomorphic* if for all $n \geq 1$ the equivalence relation \sim_n on M^n , given by $a \sim_n b$ if and only if

there exist $m, m' \in \mathcal{M}$ such that $a = m(b)$ and $b = m'(a)$, has only finitely many classes. A countable structure is called *weakly oligomorphic* if its endomorphism monoid is weakly oligomorphic.

Let us remark that weakly oligomorphic monoids are called oligomorphic in [PP16]; this leads, however, to inconsistencies with the corresponding notion for function clones, and so the definition shall henceforth be as stated here.

We first outline the idea behind our proof by recalling the situation for finite structures. When \mathbb{A} is a structure with finite domain which is not a core, then it has a non-surjective endomorphism. Restricting \mathbb{A} to the image of that endomorphism, one obtains a homomorphically equivalent structure with smaller domain. After finitely many iterations in this fashion, one obtains a structure which is a core; this structure is the core of \mathbb{A} .

When \mathbb{A} is infinite and weakly oligomorphic, one could expect the analogous argument to work, where termination of the process after finitely many steps is guaranteed by a compactness (rather than finiteness) argument using weak oligomorphicity. However, this turns out to be insufficient, which is the reason for the argument to become considerably more involved: in addition to the compactness argument, the endomorphism of \mathbb{A} yielding its model-complete core has to be generic in a sense, which is achieved via second, Fraïssé-type argument.

We start with compactness. Similarly to [BP15b], we define an equivalence relation on \mathcal{M} in order to obtain a compact object.

Definition 5.2. Extending the definition of \sim_n in Definition 5.1, we denote by \sim the equivalence relation on $\mathcal{M} \subseteq M^M$ defined by $f \sim g$ if for all $n \geq 1$ and all $x \in M^n$ we have $f(x) \sim_n g(x)$.

Lemma 5.3. *If \mathcal{M} is a topologically closed weakly oligomorphic transformation monoid, then the factor space \mathcal{M} / \sim is compact.*

Proof. The standard König's lemma argument has been executed in [BJ11] for a finer equivalence relation and monoids containing an oligomorphic permutation group, then again in [BP15b] for the case of oligomorphic function clones, and has once again been presented, perhaps more conceptually, in the most general context in [BP16b] – the proof here would be identical, so we omit it. \square

Lemma 5.4. *If \mathcal{M} is a topologically closed weakly oligomorphic transformation monoid, then \mathcal{M} contains a minimal non-empty topologically closed left ideal.*

Proof. Consider the set S of all non-empty topologically closed subsets I' of \mathcal{M} / \sim with the property that whenever $[f]_{\sim} \in I'$ and $m \in \mathcal{M}$, then $[m \circ f]_{\sim} \in I'$. Then S is non-empty and closed under finite intersections, and whence closed under arbitrary intersections by compactness. Hence, it contains a minimal element I' . The preimage of I' under the factor mapping from \mathcal{M} to \mathcal{M} / \sim is closed, left-invariant, and minimal with this property. \square

In a sense, any function in a minimal non-empty closed left ideal of \mathcal{M} as guaranteed by Lemma 5.4 can be considered to have minimal range, analogous to the finite case described above. We now argue that every minimal non-empty closed left ideal of \mathcal{M} has a generic member. Observe that weak oligomorphicity is not required for this step.

The argument is a Fraïssé-type argument (cf. [Fra54, Fra86]), but for partial functions rather than finite relational structures. There are several ways around this, for example by viewing partial functions as 2-sorted structures (as exemplified in [BPPb, Lemma 44]), or via

a more general category-theoretic approach [Kub14]. We simply avoid these technicalities by remaining vague.

For partial unary functions p, q on the domain M of \mathcal{M} , we say that p *embeds into* q if $\text{dom}(p) \subseteq \text{dom}(q)$ and the restriction of q to the domain of p is equivalent to p with respect to \sim , where \sim is extended to partial functions similarly as in Definition 5.2.

Lemma 5.5. *Let I be a minimal non-empty closed left ideal of a closed transformation monoid \mathcal{M} . Then*

$$\{f|_F \mid f \in I, F \subseteq M \text{ finite}\}$$

is a Fraïssé category under embeddings as above.

Proof. We check the amalgamation property. Consider $f|_A, g|_B \in I$. There exists $m \in \mathcal{M}$ such that $m \circ f|_{A \cup B} = g|_{A \cup B}$ since I is a minimal non-empty closed left ideal, and so $f|_A$ embeds into $g|_{A \cup B}$. So does $g|_B$, and we have an amalgam. \square

The Fraïssé-limit of the category in Lemma 5.5 is a function from M to M . Restricting \mathcal{M} to the range of such a generic function in a closed left ideal yields a model-complete core:

Lemma 5.6. *Let $\xi \in \mathcal{M}$ be the Fraïssé limit of any Fraïssé category as in Lemma 5.5, and let X be its range. Then the monoid*

$$\tilde{\mathcal{M}} := \{f \in X^X \mid \forall F \subseteq X \text{ finite } \exists m \in \mathcal{M} (f|_F = m|_F)\}$$

is a model-complete core, i.e., has dense invertibles.

Proof. By definition, $\tilde{\mathcal{M}}$ is a closed monoid. Note that $\xi \in \mathcal{M}$ since \mathcal{M} is closed. The automorphisms of ξ are in fact permutations $\alpha \in X^X$ such that $\alpha \circ \xi$ is isomorphic to ξ under our notion of embedding; that is the case if and only if $\alpha \in \tilde{\mathcal{M}}$. We claim that the automorphisms of ξ are dense in $\tilde{\mathcal{M}}$. So let $f \in \tilde{\mathcal{M}}$ and $F \subseteq X$ be finite. Then $f|_F = m|_F$ for some $m \in \mathcal{M}$. But $m|_F$ is a partial isomorphism of ξ , and hence extends to an automorphism $\alpha \in X^X$ of ξ by the homogeneity of ξ , proving our claim. \square

We can now derive Theorem 1.1 in its more general form for weakly oligomorphic structures.

Theorem 5.7. *Every weakly oligomorphic structure \mathbb{A} is homomorphically equivalent to a model-complete core \mathbb{B} . Moreover, \mathbb{B} is ω -categorical and unique up to isomorphism.*

Proof. Let $\mathcal{M} := \text{End}(\mathbb{A})$, and $\xi, X, \tilde{\mathcal{M}}$ be as in Lemma 5.6. We set \mathbb{B} to be the restriction of \mathbb{A} to X . Clearly, \mathbb{A} and \mathbb{B} are homomorphically equivalent. In order to see that \mathbb{B} is a model-complete core, we show that $\text{End}(\mathbb{B}) = \tilde{\mathcal{M}}$. The only non-trivial inclusion is $\text{End}(\mathbb{B}) \subseteq \tilde{\mathcal{M}}$. So let $e \in \text{End}(\mathbb{B})$, and let x be a finite tuple of elements in X ; we show that there exists an element of $\tilde{\mathcal{M}}$ which agrees with e on x . Let y be so that $x = \xi(y)$. Denoting the invertibles of $\tilde{\mathcal{M}}$ by \mathcal{G} , we can pick $\alpha \in \mathcal{G}$ such that $\alpha \circ \xi(x) = x$ using the homogeneity of ξ . Then we have $e(x) = e \circ \alpha \circ \xi(x) = e \circ \alpha \circ \xi \circ \xi(y)$. Since $e \circ \alpha \circ \xi \in \mathcal{M}$, there is $m \in \mathcal{M}$ such that $m \circ (e \circ \alpha \circ \xi)(\xi(y)) = \xi(y)$, so $m \circ e \circ \alpha \circ \xi(x) = x$, hence $m \circ e(x) = x$. Replacing m by $\beta \in \mathcal{G}$ using the homogeneity of ξ , we get $e(x) = \beta^{-1}(x)$, and so \mathbb{B} is indeed a model-complete core.

Clearly, $\text{End}(\mathbb{B})$ is weakly oligomorphic, and hence it is oligomorphic since it is a model-complete core. Whence, \mathbb{B} is ω -categorical. Its uniqueness follows easily from the definitions, as in previous well-known proofs. \square

Finally, we would like to connect the concepts of *model-complete cores* and *reflections*. Let \mathcal{C} be a function clone on a set C , let D be a set, and let $f: C \rightarrow D$ and $g: D \rightarrow C$ be functions. The *reflection* of \mathcal{C} by f, g is the set

$$\{f(t(g(x_1), \dots, g(x_n))) \mid t \in \mathcal{C}\}.$$

The reflection of a transformation monoid is defined similarly [BOP]. The following can be derived directly from Lemma 5.6, without proving Theorem 5.7.

Proposition 5.8. *Let \mathcal{M} be a weakly oligomorphic closed transformation monoid. Then it has a reflection into an oligomorphic closed model-complete core $\tilde{\mathcal{M}}$, which in turn has a reflection into \mathcal{M} .*

Proof. Let ξ, X be as in Lemma 5.6. Clearly, the reflection $\{\xi \circ m|_X \mid m \in \mathcal{M}\}$ is contained in $\tilde{\mathcal{M}}$. Conversely, the reflection $\{m \circ \xi \mid m \in \tilde{\mathcal{M}}\}$ is contained in \mathcal{M} . \square

If we are not interested in obtaining Theorem 1.1, but only in its utility for CSPs, then we do not need to use Theorem 5.7, but can directly apply Proposition 5.8. A function clone is called *weakly oligomorphic* if the monoid of its unary functions is weakly oligomorphic.

Corollary 5.9. *Every weakly oligomorphic closed function clone \mathcal{C} has a reflection into an oligomorphic closed model-complete core $\tilde{\mathcal{C}}$, which in turn has a reflection into \mathcal{C} . In particular, the CSP of any weakly oligomorphic structure is polynomial-time equivalent to the CSP of an oligomorphic model-complete core.*

Proof. The proof of the first statement is identical to that of Proposition 5.8. The second statement then follows from [BOP, Proposition 4.6]. \square

REFERENCES

- [BEKP] Manuel Bodirsky, David Evans, Michael Kompatscher, and Michael Pinsker. A counterexample to the reconstruction of ω -categorical structures from their endomorphism monoids. *Israel Journal of Mathematics*. To appear. Preprint arXiv:1510.00356.
- [Ber11] Clifford Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Pure and Applied Mathematics. Taylor and Francis, 2011.
- [BG08] Manuel Bodirsky and Martin Grohe. Non-dichotomies in constraint satisfaction complexity. In Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M. Halldórsson, Anna Ingólfssdóttir, and Igor Walukiewicz, editors, *Proceedings of the International Colloquium on Automata, Languages and Programming (ICALP)*, Lecture Notes in Computer Science, pages 184–196. Springer Verlag, July 2008.
- [BJ11] Manuel Bodirsky and Markus Junker. \aleph_0 -categorical structures: interpretations and endomorphisms. *Algebra Universalis*, 64(3-4):403–417, 2011.
- [BK08] Manuel Bodirsky and Jan Kára. The complexity of equality constraint languages. *Theory of Computing Systems*, 3(2):136–158, 2008. A conference version appeared in the proceedings of Computer Science Russia (CSR’06).
- [BK09] Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. *Journal of the ACM*, 57(2):1–41, 2009. An extended abstract appeared in the Proceedings of the Symposium on Theory of Computing (STOC).
- [Bod07] Manuel Bodirsky. Cores of countably categorical structures. *Logical Methods in Computer Science*, 3(1):1–16, 2007.
- [Bod12] Manuel Bodirsky. Complexity classification in infinite-domain constraint satisfaction. Mémoire d’habilitation à diriger des recherches, Université Diderot – Paris 7. Available at arXiv:1201.0856, 2012.
- [BOP] Libor Barto, Jakub Opršal, and Michael Pinsker. The wonderland of reflections. *Israel Journal of Mathematics*. To appear. Preprint arXiv:1510.04521.

- [BP11] Manuel Bodirsky and Michael Pinsker. Reducts of Ramsey structures. *AMS Contemporary Mathematics, vol. 558 (Model Theoretic Methods in Finite Combinatorics)*, pages 489–519, 2011.
- [BP14] Manuel Bodirsky and Michael Pinsker. Minimal functions on the random graph. *Israel Journal of Mathematics*, 200(1):251–296, 2014.
- [BP15a] Manuel Bodirsky and Michael Pinsker. Schaefer’s theorem for graphs. *Journal of the ACM*, 62(3):52 pages (article number 19), 2015. A conference version appeared in the Proceedings of STOC 2011, pages 655–664.
- [BP15b] Manuel Bodirsky and Michael Pinsker. Topological Birkhoff. *Transactions of the American Mathematical Society*, 367:2527–2549, 2015.
- [BP16a] Libor Barto and Michael Pinsker. The algebraic dichotomy conjecture for infinite domain constraint satisfaction problems. In *Proceedings of LICS’16*, pages 615–622, 2016. Preprint arXiv:1602.04353.
- [BP16b] Manuel Bodirsky and Michael Pinsker. Canonical functions: a new proof via topological dynamics. Preprint arXiv:1610.09660, 2016.
- [BPPa] Manuel Bodirsky, Michael Pinsker, and András Pongrácz. Projective clone homomorphisms. *Journal of Symbolic Logic*. To appear. Preprint arXiv:1409.4601.
- [BPPb] Manuel Bodirsky, Michael Pinsker, and András Pongrácz. Reconstructing the topology of clones. *Transactions of the American Mathematical Society*. To appear. Preprint arXiv:1312.7699.
- [BS81] Stanley N. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.
- [Fra54] Roland Fraïssé. Sur l’extension aux relations de quelques propriétés des ordres. *Annales Scientifiques de l’École Normale Supérieure*, 71:363–388, 1954.
- [Fra86] Roland Fraïssé. *Theory of Relations*. Elsevier Science Ltd, North-Holland, 1986.
- [FV99] Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory. *SIAM Journal on Computing*, 28:57–104, 1999.
- [GP16] Mai Gehrke and Michael Pinsker. Uniform Birkhoff. Preprint, 2016.
- [Hod97] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.
- [KP] Michael Kompatscher and Trung Van Pham. A complexity dichotomy for poset constraint satisfaction. In *Proceedings of STACS 2017*. To appear. Preprint arXiv:1603.00082.
- [KPT05] Alexander Kechris, Vladimir Pestov, and Stevo Todorcevic. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geometric and Functional Analysis*, 15(1):106–189, 2005.
- [Kub14] Wiesław Kubiś. Fraïssé sequences: category-theoretic approach to universal homogeneous structures. *Annals of Pure and Applied Logic*, 165:1755 – 1811, 2014.
- [Mac11] Dugald Macpherson. A survey of homogeneous structures. *Discrete Mathematics*, 311(15):1599–1634, 2011.
- [PP16] Christian Pech and Maja Pech. Towards a Ryll-Nardzewski-type theorem for weakly oligomorphic structures. *Mathematical Logic Quarterly*, 62(1–2):25–34, 2016.
- [Sch15] Friedrich Martin Schneider. A uniform Birkhoff theorem. Preprint arXiv:1510.03166, 2015.
- [Tho91] Simon Thomas. Reducts of the random graph. *Journal of Symbolic Logic*, 56(1):176–181, 1991.

DEPARTMENT OF ALGEBRA, MFF UK, SOKOLOVSKA 83, 186 00 PRAHA 8, CZECH REPUBLIC

E-mail address: `libor.barto@gmail.com`

URL: `http://www.karlin.mff.cuni.cz/~barto/`

INSTITUT FÜR COMPUTERSPRACHEN, THEORY AND LOGIC GROUP, TECHNISCHE UNIVERSITÄT WIEN, FAVORITENSTRASSE 9/E1852, A-1040 WIEN, AUSTRIA

E-mail address: `michael@logic.at`

URL: `https://www.logic.at/staff/kompatscher/`

DEPARTMENT OF ALGEBRA, MFF UK, SOKOLOVSKA 83, 186 00 PRAHA 8, CZECH REPUBLIC

E-mail address: `mirek@olsak.net`

INSTITUT FÜR COMPUTERSPRACHEN, THEORY AND LOGIC GROUP, TECHNISCHE UNIVERSITÄT WIEN, FAVORITENSTRASSE 9/E1852, A-1040 WIEN, AUSTRIA

E-mail address: `pvtrung@math.ac.vn`

URL: `https://www.logic.at/staff/pvtrung/`

DEPARTMENT OF ALGEBRA, MFF UK, SOKOLOVSKA 83, 186 00 PRAHA 8, CZECH REPUBLIC

E-mail address: `marula@gmx.at`

URL: `http://dmg.tuwien.ac.at/pinsker/`